ELSEVIER

Computational Geometry 6 (1996) 195–214

Computational Geometry
Theory and Applications

Erdős distance problems in normed spaces

Peter Brass

Universität Greifswald, Fachrichtungen Mathematik-Informatik, Jahnstrasse 15a, D-17489 Greifswald, Germany

Communicated by David Avis; submitted 15 May 1994; accepted 30 May 1995

Abstract

We study the problems of the maximum numbers of unit distances, largest distances and smallest distances among n points in a two-dimensional normed space. We determine the exact maximum numbers of smallest and largest distances for each normed space, the maximum number of unit distances for each normed space in which the unit sphere is not strictly convex, and show that the best known upper bound for the euclidean case applies also for each normed space with strictly convex unit sphere, thereby partially answering a question of Erdős and Ulam. The results on smallest distances give also the exact maximum number of touching pairs among n translates of a convex set in the plane, thereby generalizing the results on the translative kissing number by Hadwiger and Grünbaum.

Dedicated to Professor Dr. H.-J. Kanold on the occasion of his eightieth birthday

1. Introduction

The history of distance counting problems in combinatorial geometry began with the 1946 paper [4] of Erdős, in which he gave bounds for the minimum number of distinct distances and the maximum number of unit distances determined by n points in the plane. Since then a variety of distances (unit, smallest, largest, jth smallest and jth largest) have been studied in a variety of situations (general position, convex position) as well as in higher dimensions. In each of these results only the case of euclidean spaces was considered. In 1984 Erdős [5] wrote:

"Ulam recently asked me the following question: Let $x_1, \ldots, x_n$ be n points in the plane. Does one get interesting combinatorial and geometric questions if one modifies the metric and asks how often we can have $d(x_i, x_j) = 1$? For example, he asked: What if we define the distance of two points as the sum of the absolute values of the differences of their coordinates? In this case I proved that if $n > 4$, $n \equiv 0 (\mathrm{mod}\, 4)$, then the maximum number of unit distances is $(n^2 + n)/4$. I hope to return to these questions later."

The only further results for more general metrics in the plane were on the number of point-pseudocircle incidences in [2] and on the chromatic number of the unit distance graph in arbitrary two-dimensional normed spaces in [3].

0925-7721/96/$15.00 © 1996 Elsevier Science B.V. All rights reserved
SSDI 0925-7721(95)00019-4

2. Results

We denote by $\lambda(\|\cdot\|)$ for each norm $\|\cdot\|$ on $\mathbb{R}^2$ the length (in the metric induced by this norm) of the longest line-segment contained in the boundary of the unit disk $\{x \mid \|x\| \leqslant 1\}$.

Lemma 1. *$0 \leqslant \lambda(\|\cdot\|) \leqslant 2$ holds for each norm $\|\cdot\|$ on $\mathbb{R}^2$, with $\lambda(\|\cdot\|) = 0$ iff the norm is strictly convex and $\lambda(\|\cdot\|) = 2$ iff the unit disk is a parallelogram.*

Let $u_{\|\cdot\|}(n)$ denote the maximum number of occurrences of the unit distance in a set of n points in the normed space $(\mathbb{R}^2, \|\cdot\|)$. Previous results for the euclidean case were obtained by Erdős [4], Józsa and Szemerédi [12], Beck and Spencer [1], Spencer et al. [15] and Clarkson et al. [2], culminating in

$$c_1 n^{1+c_2/\log\log(n)} \leqslant u_{\mathrm{eucl}}(n) \leqslant c_3 n^{4/3},$$

the lower bound from [4], the upper bound from [15] as well as [2]. We prove Theorem 1.

Theorem 1. *For $n \geqslant 11$ we have*

$$\begin{aligned} c_1 n \log n \leqslant u_{\|\cdot\|}(n) &\leqslant c_2 n^{4/3} && \text{if } \lambda(\|\cdot\|) = 0, \\ u_{\|\cdot\|}(n) &= \left\lfloor \tfrac{n^2}{4} \right\rfloor && \text{if } 0 < \lambda(\|\cdot\|) \leqslant 1, \\ u_{\|\cdot\|}(n) &= \left\lfloor \tfrac{n^2+n}{4} \right\rfloor && \text{if } 1 < \lambda(\|\cdot\|) \leqslant 2. \end{aligned}$$

Furthermore we show that the exact values of $u_{\mathrm{eucl}}(n)$ which were determined by Schade [14] for $n \leqslant 14$ give lower bounds for each $u_{\|\cdot\|}(n)$.

Let $l_{\|\cdot\|}(n)$ denote the maximum number of occurrences of the largest distance in a set of n points in the normed space $(\mathbb{R}^2, \|\cdot\|)$. For the euclidean case it is well known that $l_{\mathrm{eucl}}(n) = n$. This appears first explicitely in [4] where Erdős ascribes it to Hopf and Pannwitz [11] and Sutherland [16]. We prove Theorem 2.

Theorem 2.

$$\begin{aligned} l_{\|\cdot\|}(n) &= n && \text{if } \lambda(\|\cdot\|) = 0, \\ l_{\|\cdot\|}(n) &= \left\lfloor \tfrac{n^2}{4} \right\rfloor && \text{if } 0 < \lambda(\|\cdot\|) \leqslant 1, \\ l_{\|\cdot\|}(n) &= \left\lfloor \tfrac{n^2}{4} \right\rfloor + 1 && \text{if } 1 < \lambda(\|\cdot\|) < 2, \\ l_{\|\cdot\|}(n) &= \left\lfloor \tfrac{n^2}{4} \right\rfloor + 2 && \text{if } \lambda(\|\cdot\|) = 2. \end{aligned}$$

Let $s_{\|\cdot\|}(n)$ denote the maximum number of occurrences of the smallest distance in a set of n points in the normed space $(\mathbb{R}^2, \|\cdot\|)$. For the euclidean case Harborth [10] proved $s_{\mathrm{eucl}}(n) = \lfloor 3n - \sqrt{12n-3} \rfloor$, thereby proving a conjecture of Reutter [13]. We prove Theorem 3.

Theorem 3.

$$\begin{aligned} s_{\|\cdot\|}(n) &= \lfloor 3n - \sqrt{12n-3} \rfloor && \text{if } \lambda(\|\cdot\|) < 2, \\ s_{\|\cdot\|}(n) &= \lfloor 4n - \sqrt{28n-12} \rfloor && \text{if } \lambda(\|\cdot\|) = 2. \end{aligned}$$

Given a packing of translates of a convex set K we can replace each translate $v+K$ by $v+(K+(-K))/2$ and turn it thereby in a packing of translates of a centrally symmetric set with the same adjacency structure. Then we can interpret $(K+(-K))/2$ as the unit disk of a norm, apply Theorem 3 and obtain the corollary.

Corollary. *The maximum number of touching pairs in a packing of n translates of the convex set $K \subset \mathbb{R}^2$ is $\lfloor 3n-\sqrt{12n-3}\rfloor$, if K is not a parallelogram, and $\lfloor 4n-\sqrt{28n-12}\rfloor$, if K is a parallelogram.*

This generalizes the results on the translative kissing number by Hadwiger [8] and Grünbaum [7].

3. Proof of Theorems 1 and 2

In the case of a strictly convex norm ($\lambda(\|\cdot\|)=0$), the upper bound of Theorem 1 (for the lower bound see Section 4) follows from the work of Clarkson et al. [2], in which it was already remarked that the upper bound of $O(n^{2/3}m^{2/3})$ for the number of incidences between n points and m unit circles also holds for any system of points and pseudocircles with the properties that any two pseudocircles intersect in at most two points and that any pair of points has at most two pseudocircles incident to both of them. In the case of unit circles whose centers are these points, these conditions coincide and we have to show only that any two unit circles intersect at most twice. We have the following stronger property of the graph of unit distances.

Lemma 2. *Suppose two points P,Q have $k \geqslant 3$ common neighbours $R_1,\dots,R_k$ at distance one. Then $\lambda(\|\cdot\|)>0$ and there are three parallel lines g_1, g_2, g_3, with g_2 having distance one to g_1 and g_3, such that if $d(P,Q)<2$ then $P,Q\in g_2$ and $R_1,\dots,R_k\in g_1\cup g_3$ and if $d(P,Q)=2$ then $P\in g_1$, $Q\in g_3$ and $R_1,\dots,R_k\in g_2$.*

Fig. 1 illustrates both possibilities. This proves the upper bound of Theorem 1 in the case of a strictly convex norm.

We call a system of parallel lines each having distance one to both of its neighbours a unit parallel system. We denote by $\hat{\lambda}$ the length of the longest line-segment contained in the boundary of the unit-disk which is parallel to the unit parallel system. So $\hat{\lambda}\leqslant\lambda(\|\cdot\|)$, and there is always at least one parallel system for which $\hat{\lambda}=\lambda(\|\cdot\|)$. Furthermore, let $h(n)$ be defined by $h(4)=6$, $h(5)=8$,

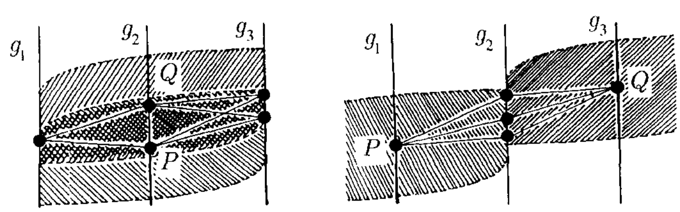

Fig. 1.

$h(n) = \lceil (n^2 - n + 13)/4 \rceil$ for $6 \leqslant n \leqslant 15$ and $h(n) = \lceil (n^2 - n + 9)/4 \rceil$ for $n \geqslant 16$. Key to the proof of Theorem 1 in the case $\lambda(\|\cdot\|) > 0$ are the following two lemmas.

Lemma 3. *The maximum number of unit distances possible in a set of $n \geqslant 4$ points that is covered by a unit parallel system is*

$$\begin{array}{ll} \lfloor \frac{1}{4} n^2 \rfloor & \text{if } 0 < \hat{\lambda} < 1, \text{ or if } \hat{\lambda} = 1 \text{ and } n \geqslant 6, \\ \lfloor \frac{1}{4}(n^2 + n) \rfloor & \text{if } 1 < \hat{\lambda} < 2, \text{ or if } \hat{\lambda} = 1 \text{ and } n = 4, 5, \text{ or if } \hat{\lambda} = 2 \text{ and } n \geqslant 11, \text{ and} \\ \lfloor \frac{1}{4}(n^2 + n) \rfloor + 1 & \text{if } \hat{\lambda} = 2 \text{ and } 4 \leqslant n \leqslant 10. \end{array}$$

Lemma 4. *Each set of n points with at least $h(n)$ unit distances between them can be covered by a unit parallel system.*

Theorem 1 now follows from Lemmas 3 and 4. We remark that there is always a set of n points with $h(n) - 1$ unit distances which cannot be covered by a unit parallel system. □

To prove the upper bound of Theorem 2 in the case of a strictly convex norm ($\lambda(\|\cdot\|) = 0$) we observe that in this case any two largest distances have a common point. For if we have two largest distances $\overline{AB}$, $\overline{CD}$ without a common endpoint, then $ABCD$ must be a convex quadrilateral. If we assume $\overline{AB}$ and $\overline{CD}$ to be a pair of opposite sides, then the diagonals $\overline{AC}$ and $\overline{BD}$ intersect in a point X with

$$\begin{aligned} \|A - B\| + \|C - D\| &< \big(\|A - X\| + \|X - B\|\big) + \big(\|C - X\| + \|X - D\|\big) \\ &= \|A - C\| + \|B - D\|, \end{aligned}$$

so $\min(\|A - B\|, \|C - D\|) < \max(\|A - C\|, \|B - D\|)$, contrary to the assumption of $\overline{AB}$, $\overline{CD}$ being largest distances.

It is an Erdős folklore theorem that any such system of segments between n endpoints with the property that any two segments have a common point contains at most n segments. For if one deletes all endpoints with only one incident segment and if there were a point P with three neighbours left, then there is a neighbour X_1 such that the other neighbours X_2, X_3 are in different halfplanes bounded by the line PX_1. But the other segment $\overline{X_1Q}$ incident to X_1 lies in one halfplane only, so it cannot intersect both $\overline{PX_2}$ and $\overline{PX_3}$.

To construct a set of $n \geqslant 3$ points with n occurences of the largest distance, we start with three points p_1, p_2, p_3 at pairwise distance one. Then we can add any further points of the circle with center p_1 and radius 1 which lie on the arc between p_2 and p_3; this does not increase the diameter, but each point gives another unit distance to p_1.

This proves Theorem 2 in the case of a strictly convex norm. For $\lambda(\|\cdot\|) > 0$, Theorem 2 is obtained by combining Lemma 4 with the following lemma.

Lemma 5. *The maximum number of unit distances possible in a set of $n \geqslant 4$ points with diameter 1 which is covered by a unit parallel system is*

$$\left\lfloor \tfrac{1}{4}n^2 \right\rfloor \quad \text{if } 0 < \hat{\lambda} \leqslant 1,$$
$$\left\lfloor \tfrac{1}{4}n^2 \right\rfloor + 1 \quad \text{if } 1 < \hat{\lambda} < 2, \quad \text{and}$$
$$\left\lfloor \tfrac{1}{4}n^2 \right\rfloor + 2 \quad \text{if } \hat{\lambda} = 2. \qquad \square$$

Proof of Lemma 1. It follows directly from the definition of $\lambda(\|\cdot\|)$ and from the triangle inequality that $0 \leqslant \lambda(\|\cdot\|) \leqslant 2$.

Whenever the unit circle contains three points of a straight line, then it contains the whole line-segment spanned by them, for if $\|x\| = \|y\| = \|\tau x + (1-\tau)y\| = 1$, $\tau \in \,]0,1[$, then we have

$$1 \geqslant \|\lambda x + (1-\lambda)y\| = \left\| \frac{1-\lambda}{1-\tau}\big(\tau x + (1-\tau)y\big) - \frac{\tau-\lambda}{1-\tau}x \right\| \geqslant \frac{1-\lambda}{1-\tau} - \frac{\tau-\lambda}{1-\tau} = 1$$

for $\lambda \in [0,\tau]$, and

$$1 \geqslant \|\lambda x + (1-\lambda)y\| = \left\| \frac{\lambda}{\tau}\big(\tau x + (1-\tau)y\big) - \frac{\lambda-\tau}{\tau}y \right\| \geqslant \frac{\lambda}{\tau} - \frac{\lambda-\tau}{\tau} = 1$$

for $\lambda \in [\tau, 1]$.

It is a well-known fact that a norm is strictly convex if and only if the boundary of the unit ball does not contain a line segment. For if it contains the line segment $\overline{ab}$, then $\|a+b\| = 2\|a/2+b/2\| = 2 = \|a\| + \|b\|$, and conversely if $\|a+b\| = \|a\| + \|b\|$, then the line $\lambda a/\|a\| + (1-\lambda)b/\|b\|$ intersects the unit circle at least for $\lambda = 0$, $\lambda = 1$ and $\lambda = \|a\|/\|a+b\|$, therefore the intersection is a line-segment.

To prove the second claim, suppose there are points a, b with $\|\lambda a + (1-\lambda)b\| = 1$ for $\lambda \in [0,1]$ and with $\|a-b\| = 2$. Then the unit circle contains the line segments $\overline{ab}$, $\overline{(-b)(-a)}$ and, since a, $(a-b)/2$ and $-b$ are collinear, also the line segments $\overline{a(-b)}$ and $\overline{(-a)b}$; it is therefore a parallelogram with vertices a, b, $-a$, $-b$. $\square$

Proof of Lemma 2. By assumption we have $\|R_i - P\| = 1$ and $\|(R_i + (P-Q)) - P\| = \|R_i - Q\| = 1$ $(i = 1, \ldots, k)$. We consider the system of parallel lines $R_i + t(P-Q)$. Each of these lines has at least two common points with the unit circle around P which have distance $\|P-Q\|$.

(a) If these parallel lines are distinct we may assume that $R_1 + t(P-Q)$ and $R_2 + t(P-Q)$ are the two outermost lines. By convexity, the parallelogram R_1, $R_1 + (P-Q)$, $R_2 + (P-Q)$, R_2 is contained in the unit disk around P. Each line $R_i + t(P-Q)$ intersects this parallelogram as well as the unit disk in a segment of length $\|P-Q\|$. Since there is at least one further line, the segments $\overline{R_1R_2}$ and $\overline{(R_1 + (P-Q))(R_2 + (P-Q))}$ must be part of the boundary of this unit disk. In this case all points R_i $(i = 1, \ldots, k)$ lie on a line which has distance 1 to P and to Q, and $\|P-Q\| = 2$.

(b) If two of these lines coincide, which we may assume to be $R_1 + t(P-Q)$ and $R_2 + t(P-Q)$, $R_2 = R_1 + \varepsilon(P-Q)$, $\varepsilon > 0$, then this common line intersects the unit circle around P in at least three points. Therefore the line segment $\overline{R_1(R_1 + (1+\varepsilon)(P-Q))}$ and by central symmetry also the line segment $\overline{(2P - R_1)(2P - R_1 - (1+\varepsilon)(P-Q))}$ are part of the boundary of the unit disk around P. By convexity the parallelogram determined by these segments is contained in the unit disk. So each line parallel to these segments and between them intersects the unit disk in a segment of length at least $(1+\varepsilon)\|P-Q\|$. Thus all remaining lines $R_i + t(P-Q)$ have to coincide with either $R_1 + t(P - Q)$ or $(2P - R_1) + t(P - Q)$. These lines have distance one to their middle parallel $P + t(P - Q)$, and we have $\|P - Q\| \leqslant 2(1+\varepsilon)^{-1}$.

This proves Lemma 2. □

Proof of Lemma 3. For each point P of a set covered by a unit parallel system all neighbours at distance one to P lie either on the same parallel (vertical neighbours: at most one above and one below P) or on the next parallel on both sides (horizontal neighbours: in those line segments of length $\hat{\lambda}$ in which the unit circle with center P touches the neighbouring parallels).

The claimed numbers of unit distances can be reached by the following constructions. For $\hat{\lambda} > 0$ there is a pair of line segments of length $\hat{\lambda}/2$ on neighbouring parallels such that each point on one segment has distance one to each point on the other segment. For $\hat{\lambda} > 1$ there is a triple of two line segments on one parallel and one line segment on a neighbouring parallel, all of length $(\hat{\lambda} - 1)/2$, such that each point on one of the two segments has distance one to one point on the other and all points of the third segment. All extremal sets for $\hat{\lambda} = 2$ and $n \leqslant 10$ are listed in Fig. 2 (by affine invariance we may assume that the unit circle is a square).

The proof of the upper bound is by induction on n. The upper bound holds for $n = 4, 5$ and $\hat{\lambda} < 1$, since unit distance triangles can occur in a set covered by a unit parallel system only if $\hat{\lambda} \geqslant 1$ (in fact for $\hat{\lambda} < 1$ the lemma directly follows from Turan's theorem). The upper bound holds for $n = 4, 5$ and $1 < \hat{\lambda} < 2$ since four points at pairwise unit distance can occur only if $\hat{\lambda} = 2$, and the graph $K_{1,2,2}$ (the unique five-vertex eight-edge K_4-free graph) is realizable with unit distances also only if $\hat{\lambda} = 2$. The upper bound can be checked for $\hat{\lambda} = 1$ and $n = 6, 7$ (in a unit parallel system with $\hat{\lambda} = 1$, each point belongs to at most one triangle on each side) as well as for $\hat{\lambda} = 2$ and $n \leqslant 12$ (using that the unit circle can be assumed to be a square).

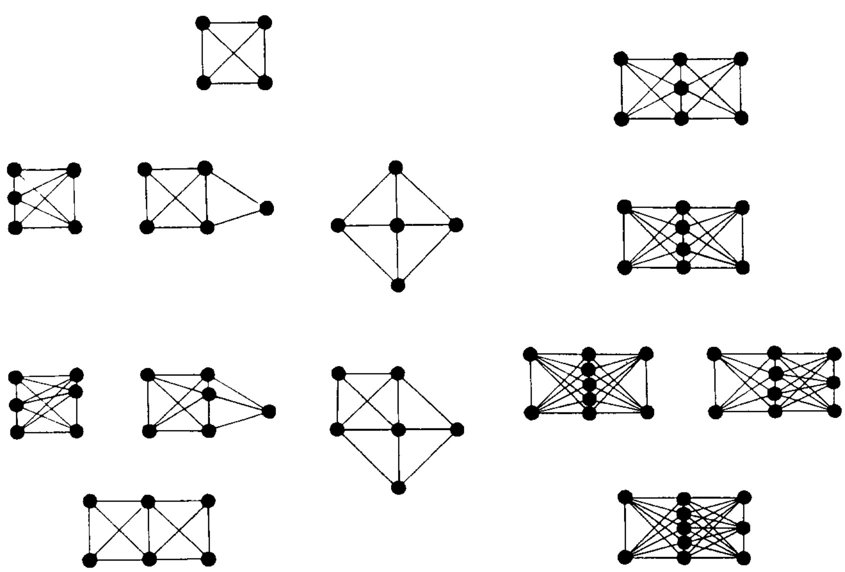

Fig. 2.

Consider now a set with maximum number of unit distances. If the set contains two points with distance larger than two, these points have no common neighbours in the graph of unit distances, so deleting both decreases the number of unit distances by at most $n-2$. By induction the set contains at most

$$\left\lfloor \tfrac{1}{4}(n-2)^2 \right\rfloor + (n-2) = \left\lfloor \tfrac{1}{4}n^2 \right\rfloor - 1 < \left\lfloor \tfrac{1}{4}n^2 \right\rfloor$$

or at most

$$\left\lfloor \tfrac{1}{4}\left((n-2)^2 + (n-2)\right) \right\rfloor + (n-2) = \left\lfloor \tfrac{1}{4}\left(n^2+n-2\right) \right\rfloor - 1 < \left\lfloor \tfrac{1}{4}\left(n^2+n\right) \right\rfloor$$

unit distances and is therefore not extremal. So we may assume that the set is covered by three consecutive parallels.

Each parallel that contains a point at all contains at least one point (the lowermost) which has at most one vertical neighbour. If the middle parallel contains at least $\lceil n/2 \rceil + 1$ points, then the lowermost point of this parallel has degree at most $\lfloor n/2 \rfloor$ and the lemma follows by induction since

$$\left\lfloor \tfrac{1}{4}n^2 \right\rfloor = \left\lfloor \tfrac{1}{4}(n-1)^2 \right\rfloor + \left\lfloor \tfrac{1}{2}n \right\rfloor \quad \text{and} \quad \left\lfloor \tfrac{1}{4}(n^2+n) \right\rfloor \geqslant \left\lfloor \tfrac{1}{4}\left((n-1)^2 + (n-1)\right) \right\rfloor + \left\lfloor \tfrac{1}{2}n \right\rfloor.$$

Again, if the outer parallels together contain at least $\lceil n/2 \rceil + 1$ points then the lowermost point of one outer parallel has degree at most $\lfloor n/2 \rfloor$ and the lemma follows by induction.

In the same way, if the middle parallel contains $\lceil n/2 \rceil$ points and one of them has no vertical neighbour, or if the outer parallels together contain $\lceil n/2 \rceil$ points and one of them has no vertical neighbour, this point has degree at most $\lfloor n/2 \rfloor$ and the lemma follows by induction.

So we may assume that either the middle parallel contains $\lceil n/2 \rceil$ points, each of them having at least one vertical neighbour, and the outer parallels contain together $\lfloor n/2 \rfloor$ points, or vice versa.

If there is a point with two vertical neighbours P, Q ($\|P-Q\| = 2$) then P and Q have at most one common neighbour on each parallel, so at most three. But more than one common neighbour is possible only for $\hat{\lambda} = 2$. Since the parallels neighbouring PQ contain together at most $\lceil n/2 \rceil$ points, there are at least $\lfloor n/2 \rfloor - 1$ points that are neighbouring neither P nor Q. Therefore P or Q has degree at most $\lfloor \lceil n/2 \rceil /2 \rfloor + 1$ in case $\hat{\lambda} < 2$ and at most $\lfloor \lceil n/2 \rceil /2 \rfloor + 2$ in case $\hat{\lambda} = 2$. In each case for $n \geqslant 6$ the lemma follows by induction.

So we may assume that each point has at most one vertical neighbour. This implies that $\lceil n/2 \rceil$ is even.

If a parallel contains at least three points and each of the points has a vertical neighbour, then this parallel contains two points with distance $t > 1$. The line segments of length $\hat{\lambda}$, in which the unit circles around these two points touch the neighbouring parallels, intersect on each parallel in a segment of length $\hat{\lambda} - t$. Therefore in case $\hat{\lambda} \leqslant 1$ these two points have no common neighbour; deleting them both loses at most $n-2$ unit distances, so the set was not extremal. This proves the lemma for $\hat{\lambda} \leqslant 1$.

Suppose now $1 < \hat{\lambda} \leqslant 2$, there is a parallel with at least three points on which each point has a vertical neighbour, and there is a point on a neighbouring parallel that also has a vertical neighbour. Then there are two points P and Q on neighbouring parallels which have distance greater than one. These two points have at most two common neighbours, one on each parallel, so removing both we lose at most n unit distances. For $\lceil n/2 \rceil$ even and $n \geqslant 5$ this implies the lemma.

It remains that either the middle parallel contains $\lceil n/2 \rceil$ points, each of them having exactly one vertical neighbour, and the outer parallels contain together $\lfloor n/2 \rfloor$ points, each having no vertical neighbour, or vice versa. In this case the maximum number of unit distances possible is $\lfloor (n^2+n)/4 \rfloor$.

This completes the proof of the lemma. □

Proof of Lemma 4. The proof is by induction on n. The lemma certainly holds for $n = 4, 5, 6$; in these cases equality is only possible if $\lambda(\|\cdot\|) = 2$ and only in the configurations shown in the first three cases of Fig. 2.

Let now a set S of n points with at least $h(n)$ unit distances be given. If there is a point $p \in S$ such that $S \setminus p$ contains at least $h(n-1)$ unit distances, then $S \setminus p$ is covered by a unit parallel system by the inductive assumption. For $n \notin \{7, 10, 14\}$ and S containing at least $h(n)$ unit distances and for $n \in \{7, 10, 14\}$ and S containing at least $h(n) + 1$ unit distances, any point of at most average degree in the graph of unit distances has this property. The exceptional cases of sets with 7, 10 or 14 points and 14, 26 or 49 unit distances, respectively, will be handled at the end.

Suppose S is not covered by this unit parallel system, so p lies between two parallels g_1 and g_2. The unit circle with center p intersects each g_i in exactly two points, so p has at most four neighbours in $S \setminus p$ at unit distance.

If $\hat{\lambda} \leqslant 1$ then $S \setminus p$ contains by Lemma 3 at most $\lfloor (n-1)^2/4 \rfloor$ unit distances, so S contains at most $\lfloor (n-1)^2/4 \rfloor + 4 < h(n)$ (for $n \geqslant 6$) unit distances, contrary to the assumption of the lemma. If $\hat{\lambda} > 1$, the same argument excludes points of degree two or less for $n \geqslant 6$.

If there is another point p' also of at most average degree then each $S \setminus p$ and $S \setminus p'$ can be covered by a unit parallel system, but not both by the same one. Thus $S \setminus \{p, p'\}$ is contained in the intersection of two unit parallel systems. This is a lattice, in which two points have distance larger than one at least if they are separated by a parallel. The points p and p' lie on parallels between lattice points and have at most four neighbours. For $n \leqslant 11$ it can be checked that none of these point sets reaches $h(n)$ unit distances; for $n \geqslant 12$ this is impossible due to the bound of Lemma 3 for the number of unit distances in $S \setminus \{p, p'\}$.

So we may assume that $\hat{\lambda} > 1$, p has degree three or four, and p is the only point with at most average degree. In this case p has two neighbours on the same parallel, which we may assume to be g_1. Let q and r be the uppermost and the lowermost point on g_1. Then $\|q - r\| \geqslant \hat{\lambda} > 1$ since the line segment in which g_1 intersects the unit disk around p has length at least $\hat{\lambda}$ (Fig. 3). So q and r each have at most one vertical neighbour, possibly p as a common neighbour and at most two further common neighbours on the parallels on both sides of g_1. Since q and r are points of degree at least $\lfloor 2h(n)/n \rfloor + 1 \geqslant n/2$, the two parallels on both sides of g_1 contain together at least $2\lfloor 2h(n)/n \rfloor - 4 \geqslant n - 6$ points. For $n \geqslant 7$ among these points there is one which has at most one vertical neighbour, which is not neighbour of p and which is not a common neighbour of q and r. This point has degree at most $n - 2\lfloor 2h(n)/n \rfloor + 3 \leqslant 5$, but it is a point of degree at least $\lfloor 2h(n)/n \rfloor + 1$. This is a contradiction for all $n \geqslant 7$.

Suppose S is a set of seven points with fourteen unit distances. If the minimum degree in the graph of unit distances is three, then the set is an extension of a set of six points with eleven unit distances. This is possible only if $\hat{\lambda} = 2$, in which case all extremal configurations are shown in the third case of Fig. 2. Any such extension of one of the four extremal configurations of six points can be covered by a unit parallel system.

If the minimum degree is four, the graph is four-regular. There are only two non-isomorphic seven-vertex four-regular graphs. The first graph of Fig. 4 contains a $K_{3,4}$ and can be seen by repeated application of Lemma 2 to be covered by a unit parallel system. This configuration is realizable as a unit distance graph if $\hat{\lambda} > 1$.

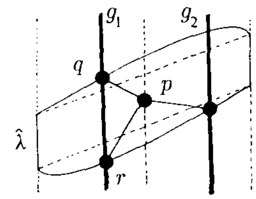

Fig. 3.

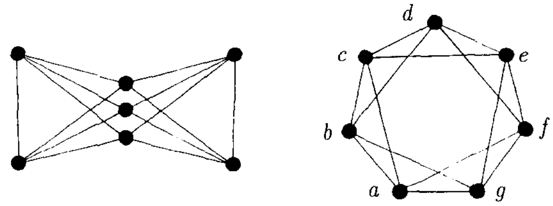

Fig. 4.

To show that the second graph of Fig. 4 is unrealizable as unit distance graph for any norm, we use the fact that the symmetry group is transitive on the complement of the graph and study two cases:

(1) There is a pair of points with distance two. We may assume $\|a-d\| = 2$. By Lemma 2 there are three consecutive lines g_1, g_2, g_3 of a unit parallel system such that $a \in g_1$, $d \in g_3$ and $b, c, f \in g_2$. Since e has distance one to d and to c, f, it must lie in the closed parallel strip bounded by g_2 and g_3. Suppose it lies properly between g_2 and g_3. Then c and f have as common neighbours d and e; this is possible only if $\hat{\lambda} = 2$ and $\|c - f\| = 2$. But e has also distance one to d; this is possible only if $e \in g_2$, contradicting the assumption. By symmetry the same argument holds also for g and the parallel strip bounded by g_1 and g_2. So $e \in g_2 \cup g_3$ and $g \in g_1 \cup g_2$, and since e and g have distance one, at least one of them, we may assume e, lies on g_2. Then g_2 contains a path $bcef$ of length three, contradicting $\|b - f\| \leqslant \|b - a\| + \|a - f\| = 2$.

(2) There is no pair of points with distance two. By Lemma 2 there are consecutive parallels g_1, g_2, g_3 of a unit parallel system such that $a, d \in g_2$ and their common neighbours $b, c, f \in g_1 \cup g_3$. Since any two of the points b, c, f have distance smaller than two, they lie on the same parallel, and we may assume $b, c, f \in g_1$. If g_1 contains additionally either e or g, it again contains a path of length three, which implies a distance three, which is impossible since any two points have a common neighbour at distance one. The same argument excludes that e, g both lie on g_2. So at least one of them, we may assume e, lies properly between g_1 and g_2. The same argument as in the first case gives a contradiction.

So the second graph cannot occur as unit distance graph, and Lemma 4 holds for each set of seven points with fourteen unit distances.

Suppose now that S is a set of ten points with 26 unit distances. Since a set of nine points contains at most 23 unit distances, the minimum degree in the graph of unit distances is at least three. If there is a point of degree three, then S is an extension of one of the two possible configurations of nine points with 23 unit distances (Fig. 2). Any such set can be covered by a unit parallel system.

If there is a point p of degree four, then $S \setminus p$ can be covered by a unit parallel system. If S is not covered by the same system then p lies between two consecutive parallels g_1 and g_2, with two neighbours on each parallel. As shown before we may assume that p is the only point with the property that $S \setminus p$ can be covered by a unit parallel system, so all other points have degree at least five. Let q_i and r_i be the uppermost and lowermost points on g_i $(i = 1, 2)$, and let g_0 and g_3 be the parallels preceeding g_1 and following g_2, respectively. Since there are only nine points in $S \setminus p$, we may assume that $g_0 \cup g_2$ contains at most four points. If $|S \cap g_0| = 0$ then q_1, r_1 have each at least three neighbours on g_2, at most one of them in common, which gives at least five points on g_2, a contradiction. If $|S \cap g_0| = 1$ then g_1 contains at least five points, at least two of them have at most one vertical neighbour and are not neighbours of p, so each of them has at least three neighbours on g_2. Since q_2 and r_2 have at most one common neighbour on g_1, this implies at least four points on g_2, again a contradiction. If $|S \cap g_0| = 2$ then g_1 contains again at least five points, since q_1 and r_1 have at most one common neighbour on g_0. Again there are two points on g_1 with at most one vertical neighbour each, and which are not neighbours of p, so each of them has at least two neighbours on g_2. Since q_2 and r_2 have at most one common neighbour on g_1, this implies at least three points on g_2, again a contradiction. Since $|S \cap g_2| \geqslant 2$ this completes the case of minimum degree four.

So we may assume that the minimum degree is five. We consider several cases.

(1) There are three points of degree five at pairwise distance one.

Removing those points leaves seven points with fourteen unit distances which is covered by a unit parallel system by the inductive assumption. Let T be the largest subset of S that is covered by the same parallel system ($7 \leqslant |T| \leqslant 10$). We may assume that $|T| \leqslant 9$.

(1.1) If $|T| = 9$ then the missing point has five neighbours at unit distance in S, so it is also covered by the same parallel system.

(1.2) If $|T| = 8$ then there are two neighbouring points p_1 and p_2, each of degree five in S, missing.

Suppose p_1 and p_2 lie in different parallel strips. In that case we may assume that g_1, g_2, g_3 are consecutive parallels such that p_i lies between g_i and g_{i+1} and has two neighbours on each of them. Since S has diameter at most two we have $T \subset g_1 \cup g_2 \cup g_3$. Let q denote a neighbour of p_1 on g_1. q has four neighbours in $g_1 \cup g_2$, and there are two further neighbours of p_1 that are not neighbours of q. So we have $|T \cap (g_1 \cup g_2)| \geqslant 7$, together with $|T \cap g_2| \geqslant 2$, a contradiction.

Suppose p_1 and p_2 both lie between the consecutive parallels g_1 and g_2. By the construction of T we may assume that p_1 and p_2 have at least one common neighbour x, which lies on g_1 or g_2. For $\hat{\lambda} = 2$ it is impossible for an equilateral triangle p_1p_2x to lie in this way in a parallel strip of width smaller than one; so we may assume $\hat{\lambda} < 2$. Let q_i and r_i be the uppermost and lowermost points on g_i. Then $\|q_i - r_i\| > \hat{\lambda}$, so q_i and r_i have no common neighbours apart from possibly p_1 and p_2. This gives at least four points on g_1 and on g_2. Exactly four points on each g_i are possible only if q_1 neighbours q_2, but this implies four points q_1, q_2, p_1, p_2 at pairwise unit distance, which is impossible for $\lambda(\|\cdot\|) \neq 2$.

(1.3) If $|T| = 7$ then T contains fourteen unit distances. T cannot contain a complete bipartite graph $K_{3,4}$, since in this case Lemma 2 would force any two points on consecutive parallels to have distance one; but this does not hold for two neighbours of a point from $S \setminus T$. By a previous result it then cannot be four-regular, so it must be an extension of one of the known sets of six points with eleven unit distances by a point of degree three. In this case we have $\hat{\lambda} = 2$. The points from $S \setminus T = \{p_1, p_2, p_3\}$ form an equilateral triangle, so they cannot all lie between the same two consecutive parallels. We may assume that g_1, g_2, g_3 are three consecutive parallels such that p_1 lies between g_1 and g_2 and p_2, p_3 lie between g_2 and g_3. p_1 has a neighbour q on g_1 which has four neighbours in $g_1 \cup g_2$. Furthermore there is a neighbour r of p_1 on g_2 that is not neighbour of q. So we have $|T \cap (g_1 \cup g_2)| \geqslant 6$, which implies that there is exactly one point x on g_3. This point x must then be a common neighbour of p_2 and p_3, which is impossible, since for $\hat{\lambda} = 2$ an equilateral triangle p_1p_2x cannot lie in a parallel strip of width smaller than one.

(2) Among any three points at pairwise distance one there is a point of degree at least six.
   (2.1) If the maximum degree is seven, we remove the unique point of degree seven and obtain a graph of nine points, nineteen edges, with seven points of degree four and two points of degree five and which does not contain a K_3. If this graph were bipartite, it had to be a $K_{4,5} - e$, but that contains a point of degree three; so the graph is not bipartite.
   (2.2) If the maximum degree is six, we remove both points of degree six and obtain a graph of eight points, fourteen or fifteen edges, with minimum degree three and which does not contain a K_3. If this graph were bipartite, it had to be a $K_{4,4} - 2e$, which can be covered by a unit parallel system by Lemma 2, but in this case it is impossible to add a point with more than four neighbours at unit distance; so the graph is not bipartite.

   In each case it is easy to see, looking at the smallest odd circle (which must be a C_5 or C_7) that such a graph does not exist.

This completes the proof in case of ten point with 26 unit distances.

Suppose finally S is a set of fourteen points with 49 unit distances. Depending on the graph of unit distances we distinguish three cases.

(1) The minimum degree is at most six.
   Since a set of 13 points contains at most 45 unit distances, the minimum degree must be at least four. If p is a point of degree at most six then $S \setminus p$ is covered by a unit parallel system by the inductive assumption. If S is not covered by the same parallel system then p has degree exactly four. Since a set of twelve points contains at most 39 unit distances, there is at most one point of degree four. So we may assume that p lies between parallels g_1 and g_2, with two neighbours on each, and all points apart from p have degree at least seven. Let q_i and r_i be the uppermost and lowermost point, respectively, on g_i. Then q_i and r_i have each at most one vertical neighbour, possibly p as a common neighbour, and at most two common neighbours on the parallels on both sides of g_i. So the parallels on both sides of g_i contain together at least eight points. This gives a total of at least seventeen points in S, a contradiction.
(2) The graph is seven-regular and bipartite.
   By repeated application of Lemma 2 the claim follows.
(3) The graph is seven-regular and not bipartite.
   By Turan's theorem the graph contains three points $\{p_1, p_2, p_3\}$ at pairwise distance one. $S \setminus \{p_1, p_2, p_3\}$ is a set of eleven points, 32 unit distances, which can be covered by a unit parallel system by the inductive assumption. p_i has degree at least five in $S \setminus \{p_{i+1}, \dots, p_3\}$, which forces it to lie on the same unit parallel system, so S is also covered by the same parallel system.

This concludes the proof of Lemma 4. □

Proof of Lemma 5. Since the set S of n points has diameter one, it lies on two consecutive parallels g_1, g_2 of the unit parallel system. $S \cap g_i$ has diameter at most one, so it contains at most one unit distance. Together with at most $\lfloor n^2/4 \rfloor$ unit distances between g_1 and g_2 (possible only if each g_i contains $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ points, and each point on g_1 has distance one to each point on g_2) this gives an upper bound of $\lfloor n^2/4 \rfloor + 2$. If this upper bound is reached, we have four points at pairwise distance one, which is possible only for $\hat{\lambda} = 2$. If we have $\lfloor n^2/4 \rfloor + 1$ unit distances, we have two points on one parallel which have distance one and which have for $n \geqslant 4$ two common neighbours on the other parallel, which is possible only for $\hat{\lambda} > 1$. So for $\hat{\lambda} > 0$ it is always possible to reach at least the claimed number of unit distances in a pointset of diameter one. □

4. Construction of sets with many unit distances

If $S, T \subset \mathbb{R}^2$ are two sets of s and t points with u_s and u_t unit distances, respectively, and if the Minkowski sum $S+T$ is direct ($s_1+t_1 \neq s_2+t_2$ for $(s_1,t_1) \neq (s_2,t_2)$) then $S+T$ is a set of st points with at least $su_t + tu_s$ unit distances (there may of course occur additional unit distances, which are not caused by this construction). The corresponding operation on the graphs is the cartesian product. This gives rise to a class of general constructions: If the unit distance graph G_S has a sufficiently big realization space (ensured in the euclidean case by rotations) such that we can select any number of realizations $S_1, S_2, \ldots$ for which the Minkowski sum $S_1+S_2+\cdots+S_k$ is direct, then we obtain in this way arbitrarily large sets with many unit distances. The simplest special case is the projected hypercube construction, which starts with G_S a single edge (so it takes the Minkowski sum of unit vectors) and gives sets of 2^k points with $k2^{k-1}$ unit distances. This is possible in any normed space. More unit distances can be obtained by taking larger starting sets (e.g., triangles), but since any construction based on direct Minkowski sums from a finite set of starting configurations gives numbers of unit distances $f(n)$ that obey the functional inequality

$$f(n) \leqslant \max_{\substack{ab=n \\ a,b\geqslant 2}} \big(af(b) + bf(a)\big)$$

for all constructible $n \geqslant n_0$ (we count of course only those unit distances that are caused by this construction, i.e., that correspond to a unit distance in one of the starting configurations), any such construction will give only $f(n) \leqslant cn\log n$ unit distances.

Furthermore, at least in the case of the projected hypercube construction, it is not possible to improve the construction by selecting 'good' subsets from larger sets obtained by the construction, since Graham [6] proved that any n vertex subgraph of an arbitrarily large cube graph contains at most $\frac{1}{2}n\log_2 n$ edges.

There is an improvement possible by taking sums that are not direct. If in the sum $S+T$ there are k pairs of points falling together, then $S+T$ is a set of $st-k$ points with at least $su_t+tu_s-2k-u_{\|\cdot\|}(k)$ unit distances. To prove this lower bound, we note that the unit distance graph G_{S+T} is obtained from the cartesian product $G_S \times G_T$ by identifying k pairs of vertices and reducing the resulting double edges. We may classify the double edges according to whether only one endpoint is identified (so they share an endpoint in $G_S \times G_T$) or both endpoints are identified. Since $G_S \times G_T$ as a cartesian product of $K_{2,3}$-free graphs is $K_{2,3}$-free, there are at most $2k$ double edges of the first kind, and since the double edges of the second kind form a unit distance graph on the k double points, there are at most $u_{\|\cdot\|}(k)$ double edges of the second kind. This proves the claimed lower bound on the number of distances.

If S is a set of integral linear combinations of a fixed set of unit vectors, v is a unit vector and the sum $S+\{0,v\}$ is not direct, then $S+\{0,v\}$ is again a set of integral linear combinations of the same unit vectors. This suggests that this class of sets deserves special study.

The sets found by Schade [14] for the euclidean case can be interpreted in this way. Take four unit vectors e_1, e_2, e_3, e_4 with $\|e_1+e_4\| = \|e_2+e_3\| = \|e_1+e_2-e_3-e_4\| = 1$; such vectors exist for each norm. Taking all 16 subset sums of $\{e_1,e_2,e_3,e_4\}$ gives a set R_{16} of 16 points with 41 unit distances. Leaving out a point of degree four, e.g., e_2+e_4, we get a set R_{15} of 15 points with 37 unit distances. Both are conjectured to be extremal and unique up to graph isomorphism. There are two sets $R_{14,1} := R_{16}\setminus\{e_2+e_4, e_1+e_3\}$ and $R_{14,2} := R_{16}\setminus\{e_1+e_3, e_1+e_3+e_4\}$ which Schade proved to be extremal (always in the euclidean case) and conjectured to be the only ones (up to graph isomorphism). For up to 13 points Schade determined all extremal sets, they are $R_{13} := R_{14,2} \setminus \{e_3\}$ (unique), $R_{12} := R_{13} \setminus \{e_3+e_4\}$ (unique), $R_{11,1} := R_{12} \setminus \{e_1\}$, $R_{11,2} := R_{13} \setminus \{e_1, e_1+e_2+e_3\}$, $R_{10} := R_{11,2} \setminus \{e_3+e_4\}$ (unique), $R_9 := R_{10} \setminus \{e_1+e_2\}$ (unique). For up to eight points there is always a subset of the triangular lattice that is extremal. A complete list of extremal sets is $R_{8,1} := \{0, e_1, e_4, 2e_1, e_1+e_4, 2e_1+e_4, 2e_1+2e_4, e_1+2e_4\}$, $R_{8,2} := \{0, e_1, e_2, e_4, e_1+e_2, e_1+e_4, e_2+e_4, e_1+e_2+e_4\}$, $R_{8,3} := R_9 \setminus \{e_1+e_2+e_3+e_4\}$, $R_7 := R_{8,1} \setminus \{2e_1\}$, $R_{6,1} := \{0, e_1, e_1+e_4, 2e_1+e_4, 2e_1+2e_4, 3e_1+2e_4\}$, $R_{6,2} := R_7 \setminus \{e_4\}$, $R_{6,3} := R_{8,1} \setminus \{e_4, 2e_4+e_1\}$, $R_{6,4} := R_{8,2} \setminus \{e_4, e_2+e_4\}$, and R_{6-i} is obtained from $R_{6,1}$ by deleting the last i points. This gives the lower bounds (exact values of the euclidean case) summarized in the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$u_{\mathrm{eucl}}(n)$	0	1	3	5	7	9	12	14	18	20	23	27	30	33

5. Proof of Theorem 3

The fundamental difference of the cases $\lambda(\|\cdot\|) < 2$ and $\lambda(\|\cdot\|) = 2$ with respect to the smallest distance is caused by the fact that for $\lambda(\|\cdot\|) < 2$ smallest distances do not intersect. For if $ABCD$ is a convex quadrilateral with diagonals AC, BD we have (as was already used in the proof of Theorem 2) $\min(\|A-B\|, \|C-D\|) \leqslant \max(\|A-C\|, \|B-D\|)$, with equality only if $\lambda(\|\cdot\|) = 2$ and the quadrilateral $ABCD$ is a circle with radius $1/2$ in the norm.

We first prove the claim in the case $\lambda(\|\cdot\|) < 2$. The proof is a direct generalization of Harborth's proof of the euclidean case [10] which is made possible by Lemma 6 below. The method was also used in his study of extremal polyominoes [9].

For each norm there is an affine image of the triangular lattice in which euclidean smallest distances are mapped on smallest distances of that norm. So each of the euclidean extremal sets, which are nearly hexagonal subsets of the triangular lattice, corresponds to a set with the same number of smallest distances with respect to the given norm. This proves the lower bound $s_{\|\cdot\|}(n) \geqslant t_\triangle(n) := \lfloor 3n - \sqrt{12n-3} \rfloor$.

The upper bound $s_{\|\cdot\|}(n) \leqslant t_\triangle(n)$ is proved by induction. It is certainly true for $n \leqslant 6$. Consider now an extremal set of n points and the polygonal subdivision of the plane generated by its graph of smallest distances. Since $s_{\|\cdot\|}(n) \geqslant t_\triangle(n) \geqslant t_\triangle(n-1) + 2 = s_{\|\cdot\|}(n-1) + 2$ the minimum degree in the graph of smallest distances is at least two. Furthermore, since $s_{\|\cdot\|}(n) \geqslant t_\triangle(n) > t_\triangle(n-m) + t_\triangle(m+1) = s_{\|\cdot\|}(n-m) + s_{\|\cdot\|}(m+1)$ for any m with $n-m, m+1 \geqslant 2$, the graph of smallest distances is 2-vertex-connected. So the graph of smallest distances has a proper boundary polygon, whose number of vertices we denote by $r = r(n)$, r_i of which are of degree i.

Definition. By an angular measure we mean a measure μ on the unit circle C_0 with center 0 which is extended in the usual translation-invariant way to measure angles elsewhere, and which has the following properties:
(1) it is normed, i.e., $\mu(C_0) = 2\pi$,
(2) it is centrally symmetric, i.e., for $X \subset C_0$ we have $\mu(X) = \mu(-X)$,
(3) for each point $p \in C_0$ we have $\mu(\{p\}) = 0$.

The interior angle sum formula holds for any such angular measure. We now use Lemma 6.

Lemma 6. *For each norm $\|\cdot\|$ on $\mathbb{R}^2$ with $\lambda(\|\cdot\|) < 2$ there is an angular measure such that each equilateral triangle is equiangular.*

It is easy to see that the condition $\lambda(\|\cdot\|) < 2$ is necessary. By the interior angle sum formula the sum of interior angles of the boundary polygon is $(r-2)\pi$. By the previous lemma any two smallest distances with a common endpoint enclose an angle of at least $\pi/3$, so the interior angle in a boundary point of degree i is at least $(i-1)\pi/3$, and we have

$$(r-2)\pi \geqslant \sum_{i\geqslant 2} \frac{\pi}{3}(i-1)r_i.$$

Removing all $r = \sum_{i\geqslant 2} r_i$ points of the boundary polygon, we loose $\sum_{i\geqslant 2}(i-1)r_i$ smallest distances and obtain a set of $n-r$ points with at most $s_{\|\cdot\|}(n-r)$ smallest distances. So we have

$$s_{\|\cdot\|}(n) \leqslant s_{\|\cdot\|}(n-r) + \sum_{i\geqslant 2}(i-1)r_i \leqslant s_{\|\cdot\|}\big(n-r(n)\big) + 3r(n) - 6.$$

Let f now denote the number of bounded faces of the polygonal subdivision of the plane defined by the graph of smallest distances. Each face is incident to at least 3 edges, each interior edge is incident to 2 faces. Counting face-edge incidences we obtain $3f \leqslant 2s_{\|\cdot\|}(n) - r$. Together with the Euler polyhedron formula $f = s_{\|\cdot\|}(n) - n + 1$ we obtain

$$r(n) \leqslant 3n - s_{\|\cdot\|}(n) - 3.$$

The claim now follows from Lemma 7.

Lemma 7. *Let $f, g : \mathbb{N} \mapsto \mathbb{N}$ be functions and a, b, c, n_0 be positive numbers such that*

$$f(n) \leqslant f\big(n - g(n)\big) + ag(n) - 2b, \qquad cg(n) \leqslant an - f(n) - b$$

holds for all $n \geqslant n_0$, where $cn_0 \geqslant 2b$.

Let furthermore $f(n) \leqslant an - \sqrt{4bcn - d}$ holds for all $n \leqslant n_0$ and some d with $0 \leqslant d \leqslant 4bc$.

Then $f(n) \leqslant an - \sqrt{4bcn - d}$ holds for all n.

To prove the theorem in the case $\lambda(\|\cdot\|) = 2$ we may again use affine invariance to assume that $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$.

We first show that $s_{\|\cdot\|}(n) \geqslant t_{\boxtimes}(n) := \lfloor 4n - \sqrt{28n - 12} \rfloor$. To construct such a set we note that a subset of the square lattice $\mathbb{Z}^2$ which is bounded by a regular octagon of side-length k (in the norm) contains $7k^2 + 4k + 1$ points which generate $28k^2 + 2k = t_{\boxtimes}(7k^2 + 4k + 1)$ smallest distances. To obtain the intermediate sets of $7k^2 + 4k + 1 + i \leqslant 7(k+1)^2 + 4(k+1) + 1$ points with $t_{\boxtimes}(7k^2 + 4k + 1 + i)$ smallest distances we add points along the boundary. Let the sides of the octagon be labeled by $1, \ldots, 8$, beginning with the leftmost side and in clockwise direction. The i points are then added to the sides in the order $8, 1, 2, 3, 4, 2, 5, 6, 4, 3, 4, 5, 6, 7$, always filling up a side until no further point of degree four can be added, before starting a new side by adding a point af degree three (Fig. 5 shows the resulting sets for $k = 2$ and $i = 1, 6, 14$ and 30). Since

$$\left\lceil \sqrt{28(7k^2 + 4k + 1 + i) - 12} \right\rceil = 14k + 4 + a$$

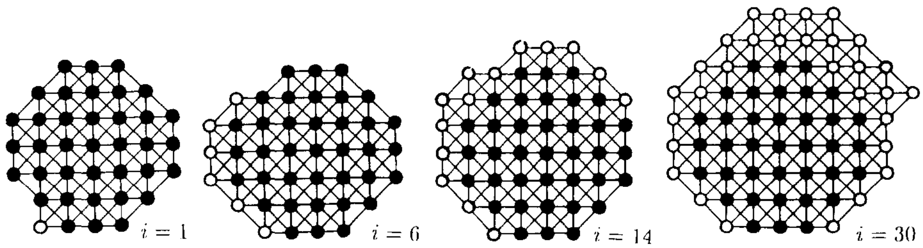

Fig. 5.

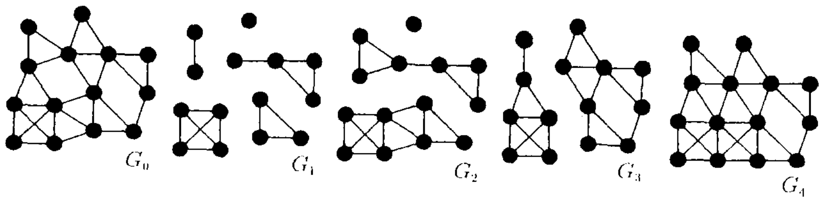

Fig. 6.

for

$$(a-1)k + \left\lfloor \frac{1}{28}(a-1)(a+7) \right\rfloor < i \leqslant ak + \left\lfloor \frac{1}{28}a(a+8) \right\rfloor,$$

and $a \leqslant 14$ (i.e., $i \leqslant 14k + 11$) these sets contain the claimed number of smallest distances.

To prove the upper bound $s_{\|\cdot\|}(n) \leqslant t_{\boxtimes}(n)$ we will first show that each extremal set S, which we take to be scaled such that the smallest distance is one, can be assumed to be a subset of the square lattice $\mathbb{Z}^2$.

We call an edge $\{(x_1, x_2), (y_1, y_2)\}$ of the graph G_0 of smallest distances of S a regular edge if $|x_1 - y_1| \in \{0, 1\}$ and $|x_2 - y_2| \in \{0, 1\}$, a horizontal edge if $|x_1 - y_1| = 1$ and $0 < |x_2 - y_2| < 1$, and a vertical edge if $0 < |x_1 - y_1| < 1$ and $|x_2 - y_2| = 1$. Each edge belongs to one of these types. Let G_1, G_2 and G_3 denote the graphs of regular, of regular and horizontal, and of regular and vertical edges, respectively (Fig. 6). In G_1, G_2 and G_3 the points of a connected component all belong to the same translate of $\mathbb{Z}^2$, of $\mathbb{Z} \times \mathbb{R}$ and of $\mathbb{R} \times \mathbb{Z}$, respectively. Suppose G_1 is disconnected. Since S is extremal, the graph of smallest distances is connected. So there are two points of S that are joined either by a horizontal or by a vertical edge, in which case they are in different connected components of G_3 or of G_2, respectively. We may assume that G_3 is disconnected. Then we can move all points of a connected component of G_3 by some vertical translation $(0, \varepsilon)$ without changing the smallest distance graph, since all edges leaving that component are horizontal edges, which admit a vertical translation of one of their endpoints, until one of these edges becomes regular (Graph G_4 in Fig. 6).

So there is a set in which there are at least as many smallest distances as in S and in which the graph of regular edges G_1 has fewer connected components. By repeated application we obtain a set in which G_1 has only one connected component, which is therefore a subset of some translate of $\mathbb{Z}^2$.

For each finite subset $S \subset \mathbb{Z}^2$ each point that is vertex of the convex hull of S has degree at most four. So $s_{\|\cdot\|}(n) \leqslant s_{\|\cdot\|}(n-1) + 4$. For $n \geqslant 6$ we have $4 \geqslant t_{\boxtimes}(n) - t_{\boxtimes}(n-1) \geqslant 3$. To prove $s_{\|\cdot\|}(n) \leqslant t_{\boxtimes}(n)$

Fig. 7.

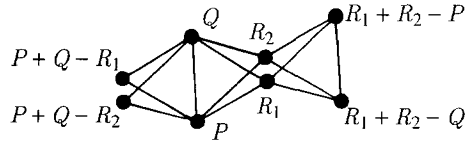

Fig. 8.

it is therefore sufficient to prove it for those n_i with $s_{\|\cdot\|}(n_i+1) \leqslant s_{\|\cdot\|}(n_i)+3$, since for $n_i - k > n_{i-1}$ we have $s_{\|\cdot\|}(n_i-k) = s_{\|\cdot\|}(n_i) - 4k = t_{\boxtimes}(n_i) - 4k \leqslant t_{\boxtimes}(n_i-k)$. We prove it by induction on the n_i. We have $s_{\|\cdot\|}(n) = t_{\boxtimes}(n)$ for $n \leqslant 6$. Let $S \subset \mathbb{Z}^2$ be now an extremal set of n_i points. If S contains a point of degree at most three then $s_{\|\cdot\|}(n_i) \leqslant s_{\|\cdot\|}(n_i-1)+3 = s_{\|\cdot\|}(n_{i-1})+3 = t_{\boxtimes}(n_{i-1})+3 \leqslant t_{\boxtimes}(n_i)$ by the inductive assumption. So we may assume that the minimum degree is four. By the definition of n_i no further point of degree at least four can be added to S. We will show that S is an octagon.

For this we first look at a diagonal edge $\overline{PQ}$ with $Q - P \in \{(1,1),\ (1,-1)\}$. P and Q each have degree at least four, so there are at least two points of S that are neighbours of P or Q in one of the open halfplanes defined by PQ. If neither of the two points with distance 1 to P and Q belonged to S, we could add one of them as a point of degree at least four; by the definition of n_i this is impossible, so each diagonal edge belongs to a triangle.

Then the boundary polygon has to be convex, since a nonconvex polygon contains a vertex of one of the types in Fig. 7, to each of which a point of degree four can be added. So the boundary polygon has to be an octagon. If there are m points inside the boundary polygon which do not belong to S, we can add all these points, and since afterwards each added point has degree eight, the number of smallest distances increases by more than $4m$, which is impossible for an extremal set. So all points inside the octagon belong to S.

Let $a_1, \ldots, a_8$ be the sidelengths of that octagon (starting form the leftmost side), and let $r := \sum_{j=1\cdots 8} a_j$ denote its circumference. Then it contains

$$n_i = \frac{1}{4}(r - a_1 - a_5 + 2)(r - a_3 - a_7 + 2) - \frac{1}{2}\sum_{i=1\cdots 4} a_{2i}^2 - \frac{1}{2}\sum_{i=1\cdots 4} a_{2i}$$

points (count by taking the surrounding rectangle and clipping triangles off the vertices) and

$$s_{\|\cdot\|}(n_i) = 4n_i - \frac{3}{2}r - \frac{1}{2}\sum_{i=1\cdots 4} a_{2i} - 4$$

smallest distances. To show $s_{\|\cdot\|}(n_i) = t_{\boxtimes}(n_i)$ we use $4n_i \geqslant s_{\|\cdot\|}(n_i) \geqslant t_{\boxtimes}(n_i)$ and compare the squared boundary deficits

$$0 \geqslant \left(4n_i - s_{\|\cdot\|}(n_i)\right)^2 - \left(4n_i - t_{\boxtimes}(n_i)\right)^2 \geqslant \left(\frac{3}{2}r + \frac{1}{2}\sum_{i=1\cdots 4} a_{2i} + 4\right)^2 - \left(\sqrt{28n_i - 12}\right)^2$$

$$= \left(\frac{9}{4}r^2 + \frac{3}{2}r\sum_{i=1\cdots 4} a_{2i} + 12r + \frac{1}{4}\left(\sum_{i=1\cdots 4} a_{2i}\right)^2 + 4\sum_{i=1\cdots 4} a_{2i} + 16\right)$$

$$- \left(7r\sum_{i=1\cdots 4} a_{2i} + 14r + 7(a_1 + a_5)(a_3 + a_7) - 14\sum_{i=1\cdots 4} a_{2i}^2 + 16\right)$$

$$= 14 \sum_{i=1\cdots 4} a_{2i}^2 - 3\Big(\sum_{i=1\cdots 4} a_{2i}\Big)^2 - \Big(\sum_{i=1\cdots 4} a_{2i}\Big)\Big(\sum_{i=1\cdots 4} a_{2i-1}\Big) + \frac{9}{4}\Big(\sum_{i=1\cdots 4} a_{2i-1}\Big)^2$$

$$- 7(a_1 + a_5)(a_3 + a_7) + 2\Big(\sum_{i=1\cdots 4} a_{2i} - \sum_{i=1\cdots 4} a_{2i-1}\Big) \geqslant -16.$$

The last inequality holds since the quadratic terms together give a positive semidefinite quadratic form, and the linear terms are bounded from below by -16 since in an extremal octagon any two non-adjacent sides differ in length by at most 2. Since the term is a difference of integer squares, each of which is at least 13^2 (for $n \geqslant 6$), this is sufficient to show $s_{\|\cdot\|}(n) = t_{\boxtimes}(n)$. $\square$

Proof of Lemma 6. Let P, Q be two points with $\|P - Q\| = 1$ and suppose that there are points R_1, R_2 on the same side of PQ such that $\|P - R_i\| = \|Q - R_i\| = 1$, i.e., there are more than two ways to complete $\overline{PQ}$ to an equilateral triangle. By Lemma 2, applied to P, Q, R_1, R_2, $(P+Q-R_1)$, $(P + Q - R_2)$ (Fig. 8) we find that R_1R_2 is parallel to PQ and all points of the segment $\overline{R_1R_2}$ have distance one to P and Q. A second application of Lemma 2 to R_1, R_2, P, Q, $(R_1 + R_2 - P)$, $(R_1 + R_2 - Q)$ shows that the unit circle (around R_2) contains a line segment parallel to PQ of length $\|(R_1 + P - R_2) - Q\| > \|P - Q\| = 1$.

So if $\lambda(\|\cdot\|) \leqslant 1$ then for each line segment of length one and each halfplane determined by that line segment there is a unique point that extends the segment to an equilateral triangle.

We first construct the measure in the case $\lambda(\|\cdot\|) \leqslant 1$. Let $\phi: [0, \infty[\to C_0$ be a parametrization of C_0 such that for all t, s with $t < s \leqslant t + \pi$ we have $|\sphericalangle\phi(t)0\phi(s)| = s - t$ in the normal euclidean angular measure. Let $h: [0,\infty[\to [0,\infty[$ be defined by $\|\phi(h(t)) - \phi(t)\| = 1$ and $t < h(t) < t + \pi$, i.e., the triangle $\triangle 0\phi(t)\phi(h(t))$ is equilateral. Then h is a continuous, strictly monotonic increasing function, and since each equilateral triangle can be extended to an affine-regular hexagon, $h(h(h(t))) = t + \pi$ (Fig. 9).

Then we define $F: [0, \infty[\to \mathbb{R}$ to be any monotonic increasing solution of the Abel functional equation $F(h(t)) = F(t) + \pi/3$. (F may be defined by an arbitrary monotonic function with $F(h(0)) = F(0) + \pi/3$ on $[0, h(0)]$, then it can be continued by the functional equation on $[0, \infty[$.) We have $F(t + \pi) = F(h(h(h(t)))) = F(t) + \pi$.

Let now an angle $\sphericalangle P0Q$ be given. There are $s, t \in [0, \infty[$, $t < s < t + 2\pi$, with $\phi(t) = P$, $\phi(s) = Q$. We define our angular measure by $|\sphericalangle P0Q|_{\|\cdot\|} := F(s) - F(t)$. This proves Lemma 6 in the case $\lambda(\|\cdot\|) \leqslant 1$.

Let now $\lambda(\|\cdot\|) > 1$. We call a line segment contained in the unit circle of a norm a long line segment if it is maximal and its length in that norm is greater than one. By the assumption there is at least one long segment.

Lemma 8. *There are at most two pairs of long segments.*

We will construct for $\lambda(\|\cdot\|) < 2$ a projection Π that maps C_0 on another centrally symmetric convex curve C_0^* (unit circle of another norm $\|\cdot\|^*$) such that C_0^* contains a pair of long segments less and such that Π maps equilateral triangles $\triangle 0AB$ with respect to $\|\cdot\|$ (A, B, $A - B \in C_0$) on equilateral triangles $\triangle 0\Pi(A)\Pi(B)$ with respect to $\|\cdot\|^*$ ($\Pi(A)$, $\Pi(B)$, $(\Pi(A) - \Pi(B)) \in C_0^*$). Then we define our measure μ for $\|\cdot\|$ by $\mu(X) := \mu^*(\Pi(X))$ where $X \subset C_0$ and μ^* is the measure for $\|\cdot\|^*$. By Lemma 8 this defines our measure for each C_0 with $\lambda(\|\cdot\|) < 2$ in at most two steps.

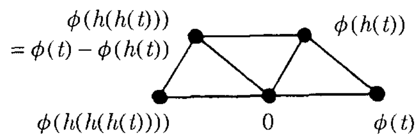

Fig. 9.

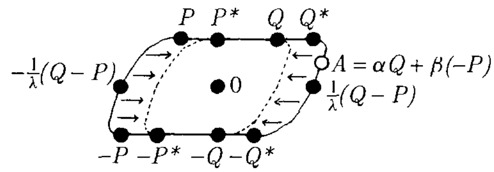

Fig. 10.

To construct Π, let $\overline{PQ} \subset C_0$ be a long segment, $\lambda := \|Q-P\| > 1$. The set $C_0 \setminus (\overline{PQ} \cup \overline{(-P)(-Q)})$ consists of two connected components M and $-M$, which we may assume to be labeled in such a way that M lies clockwise between Q and $-P$. Let $P^* := Q + (P-Q)/\lambda$ and $Q^* := P + (Q-P)/\lambda$ (Fig. 10). We define

$$\Pi(x) := \begin{cases} P^* & \text{if } x \in \overline{PP^*}, \\ x & \text{if } x \in \overline{P^*Q^*}, \\ Q^* & \text{if } x \in \overline{Q^*Q}, \\ x + (Q^* - Q) & \text{if } x \in M, \\ -\Pi(-x) & \text{if } x \in \overline{(-P)(-Q)} \cup (-M). \end{cases}$$

Then $\overline{PQ}$ is mapped on the segment $\overline{P^*Q^*}$ which has length one in $\|\cdot\|^*$ since $Q^* - P^* = (Q - P)/\lambda + Q^* - Q = \Pi((Q-P)/\lambda)$, so there is one less pair of long segments.

It remains to show that equilateral triangles are mapped on equilateral triangles. Each equilateral triangle $\triangle 0RS$ can be extended to an affine-regular hexagon $R\,S\,(S-R)\,(-R)\,(-S)\,(R-S)$ which forms together with the center 0 six equilateral triangles. By the central symmetry of Π it is sufficient to show that one of these triangles is mapped on an equilateral triangle in C_0^*. By the central symmetry of the hexagon there are three consecutive points R, S, $(S-R)$ in $\overline{PQ} \cup M$. If R and $(S-R)$ were both in $\overline{PQ}$ then $\|S\| = \|R + (S-R)\| = 2$ would follow, so there is at least one point in M.

Let $A \in M$ be a point of C_0 between Q and $Q^* - P = (Q-P)/\lambda$ and let $A =: \alpha Q + \beta(-P)$. Since A lies in the area bounded by the lines $Q(-P)$, PQ, $0(Q^* - P)$, $Q(Q^* - P)$, $(-P)(Q^* - P)$ we obtain the inequalities $\alpha + \beta \geqslant 1$, $\beta + 1 > \alpha > \beta$, $\alpha + (\lambda - 1)\beta \geqslant 1$ and $(\lambda - 1)\alpha + \beta \leqslant 1$.

The unit circle C_A intersects C_0 in a point between P^* and Q^* since

$$\begin{aligned} \|P^* - A\| &= \left\| \left(1 - \frac{1}{\alpha - \beta}\right)(-A) + \left(1 + \frac{\lambda\beta}{\alpha - \beta}\right)\frac{1}{\lambda}(P - Q) \right\| \geqslant \left|1 + \frac{\lambda\beta}{\alpha - \beta}\right| - \left|1 - \frac{1}{\alpha - \beta}\right| \\ &= 2 + \frac{\lambda\beta - 1}{\alpha - \beta} = 1 + \frac{\alpha + (\lambda - 1)\beta - 1}{\alpha - \beta} \geqslant 1 \end{aligned}$$

and

$$\begin{aligned} \|Q^* - A\| &= \left\| (1 + \beta - \alpha)P + (\alpha\lambda - 1)\frac{1}{\lambda}(P - Q) \right\| \leqslant |1 + \beta - \alpha| + |\alpha\lambda - 1| \\ &= \beta + (\lambda - 1)\alpha \leqslant 1. \end{aligned}$$

Another point of intersection is between A and $-P$ since the line $(A - Q) + t(Q - P)$ $(t \in \mathbb{R})$ is tangential to C_A and separates A and $-P$. Since the unit circles around two points with distance less than two intersect either in two points or in two parallel line segments there are no further points of intersection of C_A and C_0.

So any two points $x \in M$ and $y \in -M$ have distance $\|x-y\| > 1$. Therefore the six points of the hexagon can not all be contained in $M \cup (-M)$, so there is an equilateral triangle $0BA$ such that $B \in \overline{PQ}$ and $A \in M$. Then by the previous argument either $A = (Q-P)/\lambda$, in which case $B \in \overline{Q^*Q}$, $B - A \in \overline{PP^*}$, $\Pi(A) = (Q-P)/\lambda + Q^* - Q$, $\Pi(B) = Q^*$ and $\Pi(B) - \Pi(A) = Q - (Q-P)/\lambda = P^* = \Pi(B-A)$, or A lies between Q and $(Q-P)/\lambda$, in which case $B \in \overline{P^*Q^*}$, $A - B \in M$, $\Pi(A) = A + (Q^* - Q)$, $\Pi(B) = B$ and $\Pi(A) - \Pi(B) = A - B + (Q^* - Q) = \Pi(A-B)$. In each case the image triangle is equilateral as claimed.

This completes the proof of Lemma 6. □

Proof of Lemma 7. The proof is by induction on n. We have for $n \geqslant n_0$

$$f(n) \leqslant f\big(n - g(n)\big) + ag(n) - 2b \leqslant an - ag(n) - \sqrt{4bc\big(n - g(n)\big) - d} + ag(n) - 2b$$

by the inductive assumption. We now use the inequality on $g(n)$ in the form $f(n) - (a-c)n + b \leqslant c(n - g(n))$, where the left side is positive for $f(n) \geqslant an - \sqrt{4bcn - d}$, and obtain, isolating the square root,

$$\sqrt{4b\big(f(n) - (a-c)n + b\big) - d} \leqslant an - f(n) - 2b.$$

Both terms are positive; taking squares on both sides and reordering the terms we get the quadratic inequality

$$\big(f(n)\big)^2 - 2anf(n) + a^2n^2 - 4bcn + d \geqslant 0,$$

which implies either $f(n) \geqslant an + \sqrt{4bcn - d}$ or $f(n) \leqslant an - \sqrt{4bcn - d}$. Since $0 < cg(n) \leqslant an - b - f(n)$ we have $f(n) < an$ and the lemma follows. □

Proof of Lemma 8. Suppose that there is a norm $\|\cdot\|$ such that its unit circle C_0 contains at least three pairs of long segments. Then there is a centrally symmetric hexagon H whose sides are translates of those three pairs of long segments and which is contained in the unit disk of $\|\cdot\|$. Therefore in the norm defined by H each side of H has length greater than one. Let P_i be the vertices of H in clockwise direction, $P_{i+3} = -P_i$. Since

$$\sum_{i=1\cdots 6} |\sphericalangle P_{i-1}P_iP_{i+1}| = \sum_{i=1\cdots 6} |\sphericalangle P_{i-1}P_i0| + \sum_{i=1\cdots 6} |\sphericalangle 0P_iP_{i+1}| = 4\pi,$$

there is an i such that $|\sphericalangle P_{i-1}P_iP_{i+1}| \geqslant \pi - |\sphericalangle 0P_{i-1}P_i|$ or $|\sphericalangle P_{i-1}P_iP_{i+1}| \geqslant \pi - |\sphericalangle P_iP_{i+1}0|$. We may assume $|\sphericalangle P_1P_2P_3| \geqslant \pi - |\sphericalangle 0P_1P_2|$. Since $|\sphericalangle P_1P_2(P_2 - P_1)| \geqslant \pi - |\sphericalangle 0P_1P_2|$ and $P_2 - P_1$ lies outside H by the assumption ($\|P_2 - P_1\|_H > 1$ since $\overline{P_1P_2}$ is a long segment), $\overline{P_2(P_2 - P_1)}$ intersects $\overline{P_3(-P_1)}$. Then $P_3 =: \alpha P_2 + \beta(P_2 - P_1)$ lies in the area bounded by $\overline{P_2(P_2 - P_1)}$, $\overline{P_1P_2}$, $\overline{0(P_2 - P_1)}$ and $\overline{(-P_1)(P_2 - P_1)}$, giving rise to the inequalities $\alpha + \beta \geqslant 1$, $0 \leqslant \alpha \leqslant 1$ and $\beta \leqslant 1$. Then

$$\|P_3 - P_2\|_H = \big\|(\alpha - 1 + \beta)(P_2 - P_1) + (1-\alpha)(-P_1)\big\|_H \leqslant |\alpha + \beta - 1| + |1 - \alpha| = \beta \leqslant 1,$$

a contradiction, since $\overline{P_2P_3}$ was assumed to be a long segment. □

6. Final remarks

We determined the exact maximum numbers of smallest and of largest distances among n points for each two-dimensional real normed space and the maximum number of unit distances for those normed spaces that are not strictly convex. We conjecture that for any strictly convex norm we have in fact $u_{\|\cdot\|}(n) = \mathrm{O}(n^{1+\varepsilon})$ for all $\varepsilon > 0$. Since the Erdős lattice point construction for sets with many unit distances in the euclidean case does not generalize for other strictly convex norms, it seems possible that there are strictly convex norms with much smaller $u_{\|\cdot\|}(n)$, perhaps even $u_{\|\cdot\|}(n) = \mathrm{O}(n \log n)$. Since the sets constructed by Schade [14] all consist of integral linear combinations of unit vectors, it may be useful to study this class of constructions.

An interesting generalization of the maximum number of smallest distances is the problem of packing flexible disks, i.e., given a set of n points in the plane with smallest distance one, how many distances between 1 and $1+\varepsilon$ are there? Perhaps there is a combinatorial generalization of our concept of angular measure. There is no corresponding problem for the maximum number of almost largest distances, since a set of n points with diameter one contains at most the Turan number $\mathrm{ex}(n, K_4)$ distances between $1-\varepsilon$ and 1.

References

[1] J. Beck and J. Spencer, Unit distances, J. Combin. Theory Ser. A 37 (1984) 231–238.
[2] K.L. Clarkson, H. Edelsbrunner, L.J. Guibas, M. Sharir and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, Discrete Comput. Geom. 5 (1990) 99–160.
[3] K.B. Chilakamarri, Unit distance graphs in Minkowski metric spaces, Geom. Dedicata 37 (1991) 345–356.
[4] P. Erdős, On sets of distances of n points, Amer. Math. Monthly 53 (1946) 248–250.
[5] P. Erdős, Problems and results in combinatorial geometry, in: J.E. Goodman et al., eds., Discrete Geometry and Convexity, Ann. New York Acad. Sci. 440 (1985) 1–11.
[6] R.L. Graham, On primitive graphs and optimal vertex assignments, Ann. New York Acad. Sci. 175 (1970) 170–186.
[7] B. Grünbaum, On a conjecture of H. Hadwiger, Pacific J. Math. 11 (1961) 215–219.
[8] H. Hadwiger, Über Treffanzahlen bei translationsgleichen Eikörpern, Arch. Math. 8 (1957) 212–213.
[9] F. Harary and H. Harborth, Extremal animals, J. Combin. Inform. System Sci. 1 (1976) 1–8.
[10] H. Harborth, Lösung zu Problem 664A, Elem. Math. 29 (1974) 14–15.
[11] H. Hopf and E. Pannwitz, Problem 167, Jahresber. Deutsch. Math.-Verein. 43 (1934) 2. Abt. 114.
[12] S. Józsa and E. Szemerédi, The number of unit distances in the plane, in: Hajnal et al., eds., Infinite and Finite Sets (North-Holland, Amsterdam, 1975) 939–950.
[13] O. Reutter, Problem 664A, Elem. Math. 27 (1972) 19.
[14] C. Schade, Exakte Maximalzahlen gleicher Abstände, Diplomarbeit, Universität Braunschweig (1993).
[15] J. Spencer, E. Szemerédi and W. Trotter, Unit distances in the euclidean plane, in: B. Bollobás, ed., Graph Theory and Combinatorics (Academic Press, London, 1984) 293–304.
[16] J.W. Sutherland, Lösung der Aufgabe 167, Jahresber. Deutsch. Math.-Verein. 45 (1935) 2. Abt. 33–35.